数理解析研究所講究録
951 巻 1996 年 155-160

Boundary value problems for energy minimizing harmonic maps

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1 Introduction

We wish to consider the regularity of maps $u: M \supset \Omega \to N$ between Riemannian manifolds which are energy minimizing amongst maps satisfying a partially free boundary condition $u(\Sigma_1) \subset \Gamma$ and a fixed (Dirichlet) boundary condition $u|_{\Sigma_2} = \gamma$. Here M is a Riemannian manifold (without boundary) of dimension $m \geq 3$, Ω is a connected open subset of M with boundary $\partial\Omega$, and Σ_1 and Σ_2 are disjoint, non–empty, relatively open subsets of $\partial\Omega$ – termed the *free boundary* and the *fixed boundary* – with $\partial\Omega\cap\Sigma_1$ and $\partial\Omega\cap\Sigma_2$ of class C^2, and such that each point $x_0 \in \overline{\Sigma}_1 \cap \overline{\Sigma}_2$ admits an open neighbourhood Y such that $\overline{\Sigma}_1 \cap Y$ and $\overline{\Sigma}_2 \cap Y$ are C^2 manifolds with the common boundary $\overline{\Sigma}_1 \cap \overline{\Sigma}_2 \cap Y$. Further, N is a compact Riemannian manifold of dimension n which is isometrically embedded in $\mathbb{R}^{n+k}$ for some $k \geq 0$, and Γ is a closed submanifold (possibly with boundary) of $\mathbb{R}^{n+k}$ of dimension d, $0 \leq d \leq n$, with $\Gamma \subset N$. The submanifold Γ is termed the *supporting manifold* of the free boundary.

In this setting we can define the *energy*

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dvol \ ,$$

where $|\nabla u|^2 := \sum_{i=1}^{n+k} |\nabla u^i|^2$, for any map u in the class $H^{1,2}(\Omega, N)$, where $H^{1,2}(\Omega, N) := H^{1,2}(\Omega, \mathbb{R}^{n+k}) \cap \{u : u(x) \in N \text{ for almost all } x\}$. In this context the boundary condition $u(\Sigma_1) \subset \Gamma$ is to be understood in the trace sense. We consider boundary values $\gamma \in C^{0,1}(\overline{\Sigma}_2, N)$ which satisfy the compatibility condition $\gamma(\overline{\Sigma}_1 \cap \overline{\Sigma}_2) \subset \Gamma$.

A map $u \in H^{1,2}(\Omega, N)$ is *locally energy minimizing* with respect to the free boundary condition $u(\Sigma_1) \subset \Gamma$ and the fixed boundary condition $u|_{\Sigma_2} = \gamma$ if there exists an open covering $\mathcal{C}$ of $\overline{\Omega}$ such that $E(u) \leq E(v)$ for every $v \in H^{1,2}(\Omega, N)$ which satisfies $v(\Sigma_1) \subset \Gamma$ and $v|_{\Sigma_2} = \gamma$, and which coincides with u outside X, for some $X \in \mathcal{C}$.

In general such energy minimizing maps need not be continuous; for example by elementary degree theory any map $u \in H^{1,2}(B^m, S^{m-1})$, $m \geq 2$ satisfying the fixed boundary condition $u|_{\partial B^m} = \gamma$, with γ smooth and of non-zero degree, cannot be continuous: in particular the energy minimizer among all such maps is discontinuous. (Note that the class of $H^{1,2}$ maps satisfying the above boundary condition is always non-empty: consider $u(x) = \gamma(\frac{x}{|x|})$.) In order to discuss the regularity of such energy minimizing maps, therefore, we consider the notion of *partial regularity.* We define $x \in \overline{\Omega}$ to be a *regular point* of u, i.e. $x \in \mathrm{Reg}(u)$, if u is continuous in some (relative) neighbourhood of x. The singular set of u, $\mathrm{Sing}(u)$, is then defined to be the complement of $\mathrm{Reg}(u)$ (in $\overline{\Omega}$). The aim of partial regularity theory for harmonic maps is to obtain optimal estimates on the Hausdorff dimension ($\mathcal{H} - \dim$) of this singular set.

The first result bounding the size of the singular set of an energy minimizing map u was an interior partial regularity result: Schoen and Uhlenbeck [SU1] showed

$$\mathcal{H} - \dim(\Omega \cap \mathrm{Sing}(u)) \leq m-3, \text{ and } \mathrm{Sing}(u) \text{ is discrete in } \Omega \text{ if } m=3.$$

Well-known examples of singular energy minimizing maps demonstrate the optimality of this result.

Schoen and Uhlenbeck [SU2] also proved complete regularity at the fixed boundary for sufficiently regular Dirichlet boundary data. In terms of the current problem setting, this says:

$$\Sigma_2 \cap \mathrm{Sing}(u) = \emptyset .$$

Regularity at the free boundary for the case $\partial\Gamma = \emptyset$ was considered independently by Duzaar and Steffen [DS1, 2] and Hardt and Lin [HL2]. The authors demonstrated that, for u locally energy minimizing with respect to the free boundary condition $u(\Sigma_1) \subset \Gamma$,

$$\mathcal{H} - \dim(\mathrm{Sing}(u) \cap \Sigma_1) \leq m-3, \text{ and } \mathrm{Sing}(u) \cap \Sigma_1 \text{ is discrete (in } \Sigma_1) \text{ if } m=3.$$

Again, examples (see [DS1], [HL2]) show that this estimate is optimal. We also mention earlier works on regularity of harmonic maps at a free boundary, all of which assume that the image of u is bounded away from the focal set of Γ in N; the reader should consult the work of Gulliver and Jost [GuJ] (note that these authors only require that u be stationary for the energy functional), and earlier treatments of special cases by Wood [EL, section 3.19], Hamilton [H] and Baldes [B].

In the remainder of this paper, we describe two extensions of these results due to Frank Duzaar and the author ([DG1, 2]), and give an example of a particular singular harmonic map whose existence is confirmed by the results of [DG2].

2 Supporting manifold with boundary

We firstly wish to consider the extension of the results of the previous section to the case where $\partial\Gamma$ is non-empty. Our smoothness assumptions on the target manifold, N, and the supporting manifold for the free boundary values, Γ, are that N is a compact C^2-submanifold of $\mathbb{R}^{n+k}$, and that $\Gamma \subset N$ is a closed C^2-submanifold with boundary of $\mathbb{R}^{n+k}$. We assume that the domain $\Omega \subset M$ is a connected open subset of a Riemannian manifold without boundary of dimension $m \geq 3$, and that Σ_1 is a non-empty, relatively open subset of $\partial\Omega$. One can then show (see [DG1, Theorem]):

2.1 Theorem. *If $u \in H^{1,2}(\Omega, N)$ is locally energy minimizing on $\Omega \cup \Sigma_1$ with respect to the free boundary condition $u(\Sigma_1) \subset \Gamma$, and $\partial\Gamma \neq \emptyset$, then*

$$\mathcal{H} - \dim(\Sigma_1 \cap \operatorname{Sing}(u)) \leq m - 3,$$

and

$$\Sigma_1 \cap \operatorname{Sing}(u) \textit{ is discrete in } \Omega \cup \Sigma_1 \textit{ if } m = 3.$$

Note that regularity at the free boundary is well understood for classical minimal surfaces (i.e. the case $m = 2$); see the works of Hildebrandt and Nitsche [HN1, 2], and also the monograph of Dierkes, Hildebrandt, Küster and Wohlrab [DHKW, Section 7.7].

We wish to comment briefly on the proof of Theorem 2.1. One cannot simply generalize the methods of [DS1, 2] and [HL2] to the current situation: the fact that $\partial\Gamma$ is non-empty makes it impossible to apply their reflection arguments. This is because the natural boundary condition

$$\partial_\nu u(x) \perp \operatorname*{Tan}_{u(x)} \Gamma$$

(where here ν is the exterior unit normal vector field) is in general not satisfied, even in a weak sense, in the current situation.

The proof in [DG1] follows the scheme of the proof of partial regularity (in the interior case) of Luckhaus, [L]. In our situation the main difficulty arises in the construction of suitable comparison mappings satisfying the free boundary condition. Having done this, we can obtain a compactness theorem for energy minimizing maps, which in turn can be applied to obtain a small energy-regularity (so-called ε-regularity) theorem. This theorem states, roughly, that if

$$\liminf_{\rho \searrow 0} \rho^{2-m} \int_{B_\rho(x_0)} |\nabla u|^2 \, dx = 0$$

then u is Hölder continuous near $x_0 \in \Sigma_1$; here $B_\rho(x_0)$ is a metric ball in $\Omega \cup \Sigma_1$. The previous results, in particular the ε-regularity theorem, are then used to deduce an initial estimate on the size of the singular set; this is improved to the optimal estimate by the process of dimension-reduction. In these discussions we follow the reasoning applied in the interior situation by Simon [S]: as well as leading to the optimal dimension estimate, this approach yields more information on the make-up of the singular set (see [DG1, Theorem 4.8]).

3 A mixed boundary value problem

In [DG2] we consider regularity at the interface of the fixed and free boundary. In addition to the assumptions of section 2, we assume that Σ_2 is a non-empty, relatively open subset of $\partial\Omega$ disjoint from Σ_1, and that $\Sigma_1 \cap \partial\Omega$ and $\Sigma_2 \cap \partial\Omega$ are hypersurfaces of class C^2. Our domain is further assumed to be an *edged domain*, i.e. we assume that given $x_0 \in \overline{\Sigma}_1 \cap \overline{\Sigma}_2$, there exists an open neighbourhood Y of x_0 in M such that $\overline{\Sigma}_1 \cap Y$ and $\overline{\Sigma}_2 \cap Y$ are hypersurfaces of class C^2 with the common boundary $\overline{\Sigma}_1 \cap \overline{\Sigma}_2 \cap Y$, and that $\overline{\Sigma}_1 \cap Y$ and $\overline{\Sigma}_2 \cap Y$ intersect perpendicularly along this $(m-2)$–dimensional edge. Finally, we assume that $\gamma \in C^{0,1}(\overline{\Sigma}_2, N)$, with $\gamma(\overline{\Sigma}_1 \cap \overline{\Sigma}_2) \subset \Gamma$. We can now state the main result of [DG2] ([DG2, Theorem 1.1]):

3.1 Theorem. *Let M, Ω, Σ_1, Σ_2, N and γ fulfill the above assumptions, and let $u \in H^{1,2}(\Omega, N)$ be locally energy minimizing with respect to the free boundary condition $u(\Sigma_1) \subset \Gamma$ and the fixed boundary condition $u|_{\Sigma_2} = \gamma$. Then*

$$\overline{\Sigma}_1 \cap \overline{\Sigma}_2 \subset \mathrm{Reg}(u)\,.$$

As in section 2, the case of regularity at corners for classical minimal surfaces (i.e. the case $m = 2$) is well understood; see Grüter [G, Section 3], Grüter, Hildebrandt and Nitsche [GHN], and Hildebrandt and Nitsche [HN3].

There are several examples of situations in which a harmonic map is required to have a singularity at the free boundary; see [GuJ], [DS1] and [HL2]. In some of these examples where the target manifold N was a non-euclidean space, it was further shown that singularities in fact had to occur in the (relative) interior of the free boundary. However it was not known whether this phenomenon could occur for harmonic maps into Euclidean space. Theorem 3.1 allows us to conclude that this is indeed the case in one of the settings proposed by Duzaar and Steffen: denoting

$$B := \{x \in \mathbb{R}^m : |x| < 1\}\,; \quad B^+ := \{x \in B : x_m > 0\}\,;$$
$$S := \partial B\,; \quad \text{and} \quad S^+ := \{x \in S : x_m > 0\}\,,$$

we have

3.2 Example. ([DS1, Example 2]) Consider $\Omega = B^+$, $\Sigma_1 = \{x \in B : x_m = 0\}$, $\Sigma_2 = S^+$, $\gamma = id$, $N = \mathbb{R}^m$, and $\Gamma = S^{m-2} \times \{0\} \subset S \subset \mathbb{R}^m$.

As discussed in [DS1], there exists $u \in H^{1,2}(B^+, \mathbb{R}^m)$ which is energy minimizing with respect to the conditions $u(\Sigma_1) \subset \Gamma$ and $u|_{\Sigma_2} = \gamma$; such a u is regular on $B^+ \cup S^+$, but for topological reasons cannot be continuous on all of $\overline{\Sigma}_1$. Theorem 3.1 thus shows that u is regular on $\partial\Sigma_1$ (as was conjectured in this instance in [DS1]), and so u has the desired behaviour: precisely, we have

3.3 Corollary. *For u as in Example 3.2, we have:* $\mathrm{Sing}(u) \cap \Sigma_1 \neq \emptyset$. □

References

[B] Baldes, A.: *Harmonic maps with a partially free boundary.* Manus. Math. **40** (1982), 255–275.

[DHKW] Dierkes, U., Hildebrandt, S., Küster, A. and Wohlrab, O.: *Minimal Surfaces (Vol. II).* (Springer, New York 1992).

[DG1] Duzaar, F. and Grotowski, J.F.: *Energy minimizing harmonic maps with an obstacle at the free boundary.* Manus. Math. **83** (1994), 291–314.

[DG2] Duzaar, F. and Grotowski, J.F.: *A mixed boundary value problem for energy minimizing harmonic maps.* Math. Z. **221** (1996), 153–167.

[DS1] Duzaar, F. and Steffen, K.: *A partial regularity theorem for harmonic maps at a free boundary.* Asymptotic Analysis **2** (1989), 299–343.

[DS2] Duzaar, F. and Steffen, K.: *An optimal estimate for the singular set of a harmonic map.* J. reine angew. Math. **401** (1989), 157–187.

[EL] Eells. J. and Lemaire, L.: *Another report on harmonic maps.* Bull. London Math. **20** (1988), 385–524.

[G] Grüter, M.: *The monotonicity formula in geometric measure theory, and an application to a partially free boundary problem.* Lecture Notes in Mathematics Vol. 1357 (Eds. Hildebrandt, S. and Leis, R.), Springer, New York, 1991, 238–255.

[GHN] Grüter, M., Hildebrandt, S. and Nitsche, J.C.C.: *On the boundary behavior of minimal surfaces with a free boundary which are not minima of the area.* Manus. Math. **35** (1981), 387–410.

[HN1] Hildebrandt, S. and Nitsche, J.C.C.: *Minimal surfaces with free boundary.* Acta Math. **143** (1979), 251–272.

[HN2] Hildebrandt, S. and Nitsche, J.C.C.: *Optimal boundary regularity for minimal surfaces with a free boundary.* Manus. Math. **33** (1981), 357–364.

[HN3] Hildebrandt, S. and Nitsche, J.C.C.: *Geometric properties of minimal surfaces with free boundaries.* Math. Z. **184** (1983), 497–509.

[GuJ] Gulliver, R. and Jost, J.: *Harmonic maps which solve a free-boundary problem.* J. reine angew. Math. **381** (1987), 61–89.

[H] Hamilton, R. S.: *Harmonic maps of manifolds with boundary.* Lecture Notes in Mathematics Vol. 471, Springer, New York, 1975.

[HL1] Hardt, R. and Lin, F.H.: *Maps minimizing the L^p–norm of the gradient.* Comm. Pure Appl. Math. **40** (1987), 555–588.

[HL2] Hardt, R. and Lin, F.H.: *Partially constrained boundary conditions with energy minimizing mappings.* Comm. Pure Appl. Math. **42** (1989), 309–334.

[JM] Jost, J. and Meier, M.: *Boundary regularity for minima of certain quadratic functionals.* Math. Ann. **262** (1983), 549–561.

[L] Luckhaus, S.: *Partial Hölder Continuity for minima of certain energies among maps into a Riemannian manifold.* Indiana Univ. Math. J. **37**, (1988), 349–367.

[SU1] Schoen, R. and Uhlenbeck, K.: *A regularity theorem for harmonic maps.* J. Diff. Geom. **17** (1982), 307–335 and **18** (1983), 329.

[SU2] Schoen, R. and Uhlenbeck, K.: *Boundary regularity and the Dirichlet problem for harmonic maps.* J. Diff. Geom. **18** (1983), 253–268.

[S] Simon, L.: *Singularities of geometrical variational problems.* Regional Geometry Institute lecture notes, Utah, (1992).

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